B.Sc. part III

(1) Limits and Moore-Osgood theorem

Moore-Osgood theorem

Statement :-

If $f(x,y)$ be a real valued function of two variable x, y where $(x,y) \in A \times B$ and if $a \in \bar{A}$ the closure of $\bar{A}$ $b \in \bar{B}$ the closure of B and if

(i) $\lim_{x \to a} f(x,y)$ exists for every y for which $f(x,y)$ is defined

(ii) $\lim_{y \to a} f(x,y)$ exists for every x for which $f(x,y)$ is defined and

(iii) either of the above two limits is uniform then the double limit the repeated limits exists and are all equal

proof :- since $\lim_{x \to a} f(x,y)$ exits suppose $\lim_{x \to a} f(x,y) = g(y)$ — ①

② Again $\lim_{y \to b} f(x,y)$ exists

So let $\lim_{y \to b} f(x,y) = h(x)$ — ②

Also suppose the limit ② is uniform with respect to x which means given $\varepsilon > 0$ there exists $\delta > 0$ such that

$|f(x,y_1) - f(x,y)| < \varepsilon$ — ③

for ~~which~~ all values of y and y_1 satisfying $|y-b| < \delta$ and $|y_1 - b| < \delta$ and for all x

Now we suppose $x \to a$ in ③ we have from ①

$|g(y_1) - g(y)| < \varepsilon$ — ④

for all value of y and y for which $|y-b| < \delta$ and $|y_1 - b| < \delta$

But from ④ we conclude that $\lim_{y \to b} g(y)$ exists

(3) Suppose $\lim_{y \to b} g(y) = l$ (5)

Again if we let $y_1 \to b$ in (4) we have

$|l - g(y)| < \varepsilon$ — (6)

for which $|y - b| < \delta$

Again suppose $y_1 \to b$ in (3) then in view of (2) we get

$|h(x) - f(x,y)| < \varepsilon$ — (7)

for all y for which $|y - b| < \delta$ and for all x

Now $|l - h(x)| = |l - g(y) + g(y) - f(x,y) + f(x,y) - h(x)|$

$\leq |l - g(y)| + |g(y) - f(x,y)| + |f(x,y) - h(x)|$

$< \varepsilon + \varepsilon + \varepsilon$ from (6) and (7)

$\Rightarrow \lim_{x \to a} h(x) = l$ — (8)

Now from (5) and (8) we see that

$$\lim_{y \to b} \lim_{x \to a} = \lim_{x \to a} \lim_{y \to b} f(x,y)$$

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Problem (1) The function defined by

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$$

the repeated limits exists but the double limit which is the simultaneous limit does not exists when $(x,y) \to (0,0)$

Solution :- Here

$$\lim_{y\to 0} \lim_{x\to 0} \frac{x^2y^2}{x^2y^2+(x-y)^2}$$

$$= \lim_{y\to 0} \frac{0}{0+(0-y)^2}$$

$$= \lim_{y\to 0} 0 = 0$$

and $\lim_{x\to 0} \lim_{y\to 0} \dfrac{x^2y^2}{x^2y^2+(x-\cancel{y})^2}$

$$= \lim_{x\to 0} \frac{0}{0+x^2}$$

$$= 0$$

Hence two repeated limits exists and are equal

(5) For simultaneous double limit putting $y = x$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2y^2}{x^2y^2 + (x-y)^2}$$

$$= \lim_{x \to 0} \frac{x^2x^2}{x^2x^2 + (x-x)^2}$$

$$= \lim_{x \to 0} \frac{x^4}{x^4} = 1$$

which is different from the common values of the two repeated limits

Thus $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y)$ does not exists